THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 52, No. 7, September, 1973
Printed in U.S.A.

Impulse Response of Fibers With Ring-Shaped Parabolic Index Distribution

By D. GLOGE and E. A. J. MARCATILI

(Manuscript received March 6, 1973)

The index distribution in the cross section of a multimode fiber has an important influence on the modal group velocities and, hence, on the fiber impulse response. In this paper we derive a method for the evaluation of arbitrary circular symmetric index profiles. In particular, we compute the impulse response of a fiber with a ring-shaped parabolic index profile which exhibits useful equalizing properties. The pulse spread is found to be nearly one order of magnitude smaller than that of a fiber with an equal, but abrupt, index decline from core to cladding.

I. INTRODUCTION

Multimode operation of optical fibers relaxes the fabrication tolerances, allows the use of incoherent sources, and can alleviate handling and splicing problems. Modal (group) delay differences are nearly equalized[1,2] if the core index decreases as the square of the fiber radius from a maximum at the axis (Fig. 1). A distribution of this kind is realized in the Selfoc* fiber, which was indeed reported to have very low values of differential mode delay.[3,4]

Since then, the question has been raised whether there are other index profiles which have similar equalizing effects, but are otherwise perhaps more amenable to certain fabrication techniques or have advantages with respect to splicing or bending. Although the latter part of this question is difficult to answer at this time, it is certainly possible to identify at least one profile that has quite effective equalizing properties. Imagine a slab with a square-law index distribution in transverse direction. The group velocities of all its modes are known to be nearly equal.[1] It is then plausible to expect that these properties are approximately preserved if the slab is warped in a way which results in a tube with the cross-sectional index distribution shown in

* Registered trademark of Nippon Electric Co., Ltd. and Nippon Sheet Glass Co., Ltd.

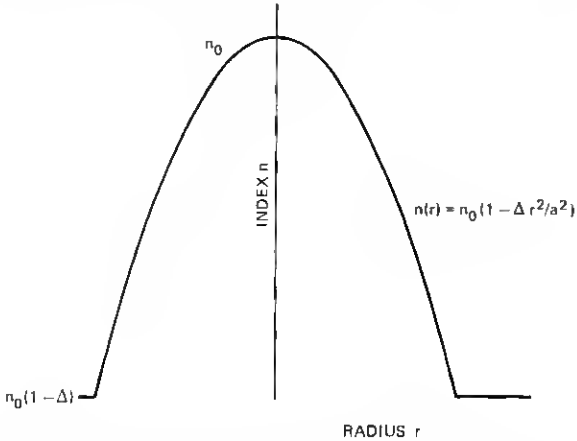

Fig. 1—Concentric parabolic index profile.

Fig. 2. Assume that a cladding material of lower refractive index fills the bore and surrounds the tube to the outside, so that a fiber is formed which guides modes within a tube-like structure with parabolic index distribution.

The purpose of this paper is to identify the modes of this structure, calculate their group velocities, and predict the impulse response to be expected when all modes propagate uncoupled and with equal power. To do this, we employ the WKB description[5] in a form which ignores the anomalies of dielectric waveguide modes near cutoff, assuming that few of all the propagating modes are close to this condition. For the sake of simplicity, we also restrict the following computations

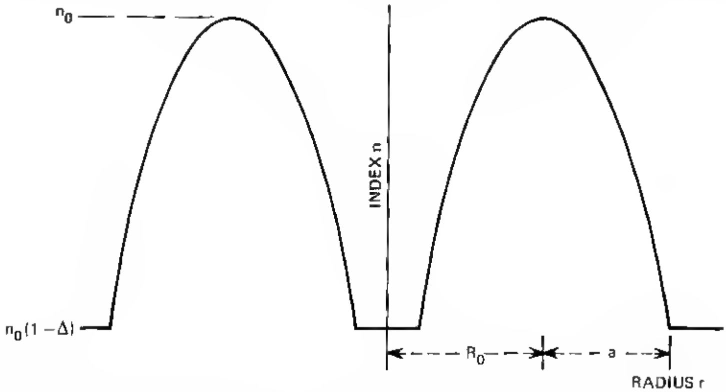

Fig. 2—Cross section through a ring-shaped parabolic index profile. Maximum index n_0 along a circle of radius R_0. Cladding index $n_0(1 - \Delta)$.

to small index variations, so that all propagation directions can be assumed paraxial to the waveguide axis and the corresponding approximations apply.

II. A CHARACTERISTIC EQUATION FOR CIRCULAR SYMMETRIC INDEX DISTRIBUTIONS

Let us adopt a cylindrical coordinate system (r, ϕ, z) and assume that the refractive index n is a function of r only. We define a local wave number

$$k(r) = 2\pi n(r)/\lambda, \tag{1}$$

where λ is the wavelength in free space. Because of the circular symmetry, we can separate the general wave equation and solve for ϕ and z. In doing so, we define an axial propagation constant β and describe the azimuthal periodicity by an azimuthal mode number ν. The remaining partial differential equation for the radial field dependence $E(r)$ has then the form

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + \left(k^2(r) - \beta^2 - \frac{\nu^2}{r^2}\right) E = 0. \tag{2}$$

Following the usual WKB approach,[5] we substitute

$$E(r) = e^{u(r)}, \tag{3}$$

ignore the second derivative $\partial^2 u/\partial r^2$, and, by solving for $\partial u/\partial r$, we obtain the solution

$$\frac{\partial u}{\partial r} = -\frac{1}{2r} \pm i\sqrt{k^2(r) - \beta^2 - \left(\nu^2 + \frac{1}{4}\right)\Big/ r^2}. \tag{4}$$

Given β and ν, we can find two radii, R_1 and R_2, at which the root in (4) vanishes (Fig. 3). These radii define a ring-shaped region within which eq. (4) has an imaginary part causing the field E to be a periodic function. Outside of the region, E decreases or increases aperiodically.

As in the 2-dimensional case,[5] decreasing (or evanescent) field characteristics outside are obtained if the total phase inside the region is

$$\int_{R_1}^{R_2} \sqrt{k^2(r) - \beta^2 - (\nu^2 + \tfrac{1}{4})/r^2}\,dr = (\mu + \tfrac{1}{4})\pi, \tag{5}$$

where μ is an integer called the meridional mode number. It determines the number of half periods of E in radial direction. The accuracy of (5) improves for large μ, but is in most cases surprisingly good even for small values of μ. Equation (5) permits an evaluation of the propaga-

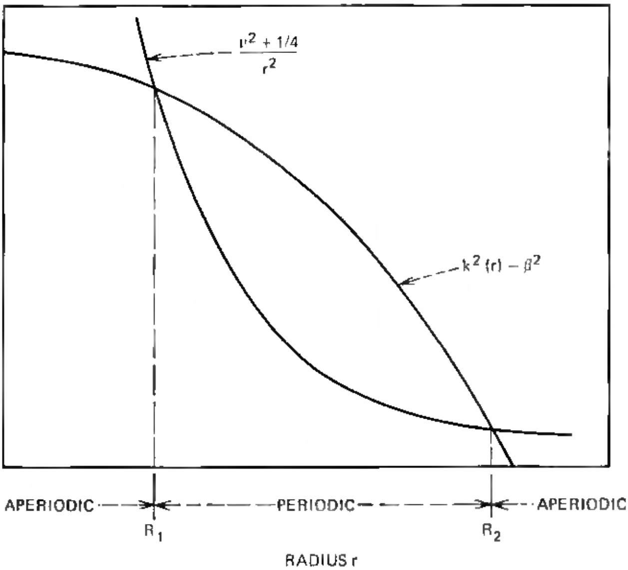

Fig. 3—Sketch defining regions of periodic and aperiodic field characteristics of a mode of azimuthal order ν.

tion constant β for given mode numbers μ and ν. This will now be done for the parabolic ring structure sketched in Fig. 2.

III. GROUP DELAY AND IMPULSE RESPONSE

Figure 2 shows a cross-sectional view of the circular symmetric index distribution. The index has a maximum value n_0 at $r = R_0$, decreases as

$$n(r) = n_0[1 - \Delta(r - R_0)^2/a^2] \qquad \text{for} \quad R - a < r < R + a, \tag{6}$$

and has a constant value

$$n(r) = n_0(1 - \Delta) \tag{7}$$

everywhere else. We assume Δ to be small compared to unity, introduce the abbreviations

$$k_0 = 2\pi n_0/\lambda \tag{8}$$

and

$$\rho = r - R_0, \tag{9}$$

and obtain, witb tbe belp of (1) and (6),

$$k^2(r) \approx k_0^2(1 - 2\Delta\rho^2/a^2). \tag{10}$$

In order to solve eq. (5) analytically, we assume in addition that $R_0 \gg a$, which permits us to replace r by R_0 in (5). As a result,

$$(\mu + \tfrac{1}{4})\pi = \int_{R_1}^{R_2} \sqrt{k_0^2 - \beta^2 - (\nu^2 + \tfrac{1}{4})/R_0^2 - 2\Delta k_0^2 \rho^2/a^2}\, dr, \quad (11)$$

which has the solution

$$\beta = [k_0^2 - (\nu^2 + \tfrac{1}{4})/R_0^2 - 2\sqrt{2\Delta}(\mu + \tfrac{1}{4})k_0/a]^{\frac{1}{2}}. \quad (12)$$

The phase constant β of a propagating mode must furthermore fulfill the condition

$$\beta \leqq k_0(1 - \Delta) \quad (13)$$

for the cladding field to have evanescent characteristics. This permits us to calculate the total number of propagating modes. Keeping ν fixed, we first determine the number of modes m in a group with the same ν. We do this by solving (12) for μ with $\beta = k_0(1 - \Delta)$. Since $\Delta \ll 1$

$$m = \mu_{\max}(\nu) + 1 = \sqrt{\Delta/2}\,ak_0 - \frac{a(\nu^2 + 1/4)}{2k_0R_0^2\sqrt{2\Delta}} + \frac{3}{4}. \quad (14)$$

This number decreases as ν increases. The largest possible ν is obtained for $m = 1$. Thus with (14)

$$\nu_{\max} = \left(2k_0^2R_0^2\Delta - \frac{k_0R_0^2}{a}\sqrt{\Delta/2}\right)^{\frac{1}{2}} - \frac{1}{4}. \quad (15)$$

For the following approximations, we ignore the terms $\frac{1}{4}$. In this case, the sum over all m from $\nu = 0$ to $\nu_{\max}$ yields

$$M = \tfrac{2}{3}ak_0^2R_0\Delta, \quad (16)$$

which is the total number of propagating modes. Using the same approximations, we can express m with the help of (15) in the form

$$m = \sqrt{\Delta/2}\,ak_0(1 - \nu^2/\nu_{\max}^2), \quad (17)$$

an expression which will be used later on.

To calculate the mode delay, we first convince ourselves with the help of (14) and (15) that the μ- and ν-terms in (12) are small (of the order Δ) compared to k_0^2. We therefore approximate β by

$$\beta = k_0 - (\nu^2 + \tfrac{1}{4})/2k_0R_0^2 - \sqrt{2\Delta}(\mu + \tfrac{1}{4})/a. \quad (18)$$

The differentiation of β with respect to the radial frequency

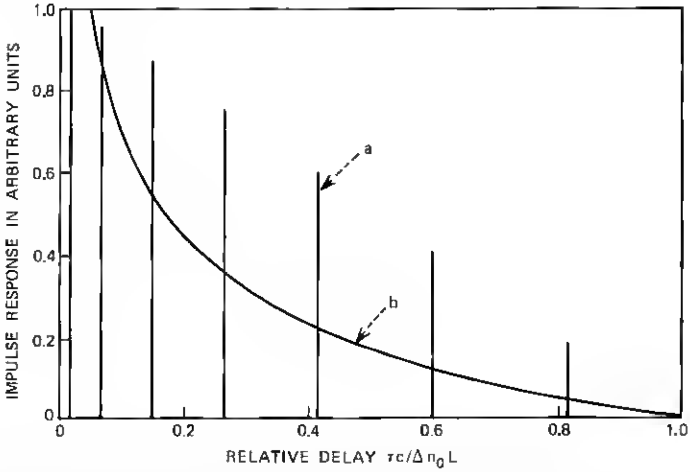

Fig. 4—Impulse response of the thin parabolic tube structure; (*a*) individual mode groups, (*b*) power distribution for large mode numbers ($R_0 \gg a$).

$\omega = ck_0/n_0$ yields the group delay

$$t = \frac{L}{c} n_0 \left[1 + \left(\nu^2 + \frac{1}{4}\right) \Big/ 2k_0^2 R_0^2\right], \tag{19}$$

where L is the fiber length and c the velocity of light in free space. Since t depends only on ν but not on μ, mode groups with the same ν have the same delay. Consequently, if all modes are excited by equal pulses of unit energy at the fiber input, the output consists of pulses of energy $m(\nu)$ delayed by $t(\nu)$. If we ignore the delay Ln_0/c common to all modes and then insert (15) into (19), again neglecting the terms $\frac{1}{4}$, we can write the delay in the form

$$\tau(\nu) = t - \frac{Ln_0}{c} = \frac{Ln_0\Delta}{c} \frac{\nu^2}{\nu_{\max}^2}. \tag{20}$$

Figure 4 illustrates the output distribution for the case in which the pulses are so narrow that individual groups are resolved. All pulses have the same (very small) width, and their heights correspond to the total energy m in each group.

More meaningful than this plot is a plot of the energy per unit time

$$p(\tau) = m d\nu/d\tau \tag{21}$$

which coincides with the power distribution in the case of very large mode numbers. We find $d\tau/d\nu$ by differentiating (20) and, if the term

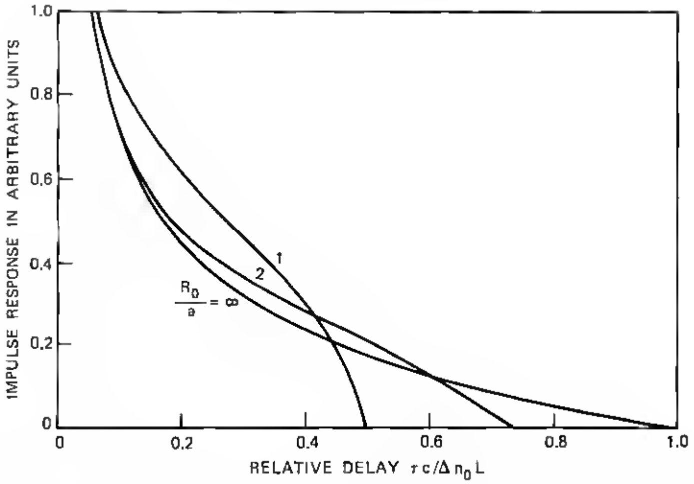

Fig. 5—Impulse response of the parabolic tube structure for various radius-to-thickness ratios.

$\frac{1}{4}$ in (14) is again ignored, we have

$$p(\tau) = \frac{caR_0k_0^2}{\sqrt{2}Ln_0}\left[\left(\frac{\Delta Ln_0}{c\tau}\right)^{\frac{1}{2}} - \left(\frac{c\tau}{\Delta Ln_0}\right)^{\frac{1}{2}}\right]. \tag{22}$$

This function is also plotted in Fig. 4. We find half the total output concentrated in the time interval

$$T = 0.12\Delta n_0 L/c. \tag{23}$$

The remaining power is drawn out in a tail of length $\Delta n_0 L/c$. This tail is caused by modes of high azimuthal order ν, modes which are not present in the 2-dimensional structure and are essentially equalized in the case of the concentric parabolic profile (Fig. 1). In this respect, the parabolic ring structure is inferior to the corresponding concentric profile, yet an effective width of 0.12 $\Delta n_0 L/c$ may be a useful improvement in comparison to a guiding structure with uniform index n_0 which theoretically produces a width $\Delta n_0 L/c$.

The condition $R_0 \gg a$ was necessary for an analytic solution of the integral (11). Exact numerical results for arbitrary ratios R_0/a are shown in Fig. 5. The corrections with respect to (22) are largest for high azimuthal orders. An exact analytical solution can only be found for the maximum delay $\tau(\nu_{\max})$ which becomes

$$\tau(\nu_{\max}) = \frac{\Delta n_0 L}{c}\left[1 - \frac{1}{8}\left(\sqrt{\frac{R_0^2}{a^2} + 8} - \frac{R_0}{a}\right)^2\right]. \tag{24}$$

As an example, consider a parabolic ring whose half width, a, is equal to the central radius R_0. Let $n_0 = 1.5$ and assume $\Delta = 1$ percent. The total width of the impulse response after 1 km of this fiber would be $\tau(\nu_{max}) = 25$ ns according to (22), but half the power is concentrated within the first 6 ns. Since high-order modes are usually lossier than the low orders, it is likely that much of the pulse tail does not reach the fiber end.

IV. CONCLUSIONS

The WKB approximation yields a simple characteristic equation for the propagating modes in fibers with arbitrary circular symmetric index distribution. We use this method to compute the impulse response of a fiber with a ring-shaped parabolic index distribution. We find that this structure has equalizing properties similar to the concentric parabolic index distribution, except for certain azimuthal mode orders, which lag behind, forming a rather long pulse tail. The rest of the power is concentrated in a time interval which, for a 1-km length and a relative index difference of 1 percent, is only 6 ns.

V. ACKNOWLEDGMENTS

Stimulating discussions with S. E. Miller and Mrs. W. Mammel's help in the numerical computation are gratefully acknowledged.

REFERENCES

1. Miller, S. E., "Light Propagation in Generalized Lens-Like Media," B.S.T.J., *44*, No. 9 (November 1965), pp. 2017–2064.
2. Kawakami S., and Nishizawa, J., "An Optical Waveguide with the Optimum Distribution of the Refractive Index with Reference to Waveform Distortion," IEEE Trans. on Microwave Theory and Tech., *MTT-16* (October 1968), pp. 814–818.
3. Uchida, M., Furukawa, M., Kitano, I., Koizumi, K., and Matsumura, H., "A Light-Focussing Fibre Guide," IEEE J. Quantum Electronics (Digest of Tech. Papers), *QE-5* (June 1969), p. 331.
4. Gloge, D., Chinnock, E. L., and Koizumi, K., "Study of Pulse Distortion in Selfoc Fibers," Electron. Letters, *8* (October 19, 1972), No. 21, pp. 526–527.
5. Morse, P. M., and Feshbach, H., *Methods of Theoretical Physics*, New York, McGraw-Hill, 1953, p. 1092.